First semestral backpaper exam 2019
B.Math. (Hons.) IInd year
Algebra III — B.Sury

Q 1. Let P be a prime ideal and suppose $P \supset I_1 \cap I_2 \cap \cdots \cap I_n$ for some ideals I_j's. Then, prove that $P \supset I_j$ for some j.

OR

Find, with proof, a maximal ideal of $\mathbb{Z}[X]$ containing 4.

Q 2. Let $\theta : \mathbf{C}[X,Y] \to \mathbf{C}[T]$ be the ring homomorphism given by $X \mapsto T^2, Y \mapsto T^3$. Prove that Ker $\theta = (X^3 - Y^2)$.

OR

Prove that if A is a Noetherian ring, then so is $A[X]$.

Q 3. Let M be a free module of rank 2 over a PID. Show that any non-zero submodule $N \neq 0$ of M is free, of rank 1 or 2. Can it be of rank 2?

OR

Let M be a left R-module over a commutative ring with unity, and let N be a submodule. If N and M/N are finitely generated, prove that M is finitely generated. Further, give an example of a free module over $\mathbb{Z}$ which has two minimal spanning sets of different cardinalities.

Q 4. Define the Jacobson radical $Jac(R)$ of a commutative ring R with unity and prove that $x \in Jac(R)$ if and only if $1 + xy$ is a unit for all $y \in R$.

Q 5. Define the companion matrix $C(f)$ of a monic polynomial $f \in K[X]$ for a field K. Prove that its characteristic polynomial is f. Further, if deg $f = 2$, prove that $C(f)$ is conjugate to its transpose by a matrix in $GL_2(K)$.